9/2/23 MATH2060B Tutorial

Announcements:

- HW2 due tomorrow 11am
- HW3 due 17/2 11 am
- after today, WEI Yunsong will teach tutorials 4-6.

($f \in C^n(I)$)

<u>Recall</u>: Taylor's Thm: let $n \in \mathbb{N}$, $I=[a,b]$, $f: I \to \mathbb{R}$ be s.t. $f', f'', \dots, f^{(n)}$ cts on I and that $f^{(n+1)}$ exists on (a,b). If $x_0 \in I$, then for any $x \in I$, there exists a point c between x and x_0 s.t.

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_n(x) \text{ Taylor Polynomial}}$$

$$+ \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{R_n(x) \text{ - remainder term in Lagrange/derivative form.}}$$

<u>Newton's Method</u>: let $I=[a,b]$, $f: I \to \mathbb{R}$ be twice differentiable on I. Sps $f(a)f(b)<0$ and that there are constants m, M s.t. $|f'(x)| \geq m > 0$, $|f''(x)| \leq M$ for $x \in I$ and let $K = \frac{M}{2m}$. Then there exists some $I^* \subseteq I$ containing a zero r of f s.t. for any $x_1 \in I^*$, the sequence (x_n) defined by

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for all } n \in \mathbb{N}$$

belongs to I^* and $x_n \to r$.

Moreover,

$$|x_{n+1} - r| \leq K |x_n - r|^2. \quad \text{for all } n \in \mathbb{N}.$$

Section 6.4

Q16: Let $I \subseteq \mathbb{R}$ be an open interval, let $f: I \to \mathbb{R}$ differentiable on I and sps $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

Give an example where RHS exists but $f''(a)$ DNE.

Hint: Use L'Hopital's Rule to establish this formula

Pf: We want to use LHR. Since $f''(a)$ exists, there is a nbhd of a where $f'(x)$ exists and is continuous at a. So $f(a+h) - 2f(a) + f(a-h)$ is differentiable for h small enough. h^2 differentiable.

$\lim_{h\to 0} h^2 = 0$. $\lim_{h\to 0} f(a+h) - 2f(a) + f(a-h) = 0$.

$(h^2)' \neq 0$ for $h>0$.

So we can apply LHR (differentiating w.r.t. h) to get:

$$\text{RHS} = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h}$$

$$= \lim_{h\to 0} \frac{f'(a+h) - f'(a) + f'(a) - f'(a-h)}{2h} = f''(a)$$

by defn of $f''(a)$.

Example: $f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$ take $a=0$.

$f'(x) = 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right)$. But $f''(0) = \lim_{x\to 0} \frac{3x^2\sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right)}{x} = \text{DNE}$.

oscillation

But $\lim_{h\to 0} \frac{f(h) - \cancel{2f(0)}^{=0} + f(-h)}{h^2} = \lim_{h\to 0} \frac{h^3\sin(\frac{1}{h}) + (-h)^3\sin(\frac{1}{-h})}{h^2}$

$= \lim_{h\to 0} \frac{2h^{\cancel{3}}\sin(\frac{1}{h})}{\cancel{h^2}} = \lim_{h\to 0} 2h\sin(\frac{1}{h}) = 0.$

Q18: Let f, g defined on I, $c \in I$, and that $f^{(n)}, g^{(n)}$ exist and are cts on I.
Sps $f^{(1)}(c), \ldots, f^{(n-1)}(c), g^{(1)}(c), \ldots, g^{(n-1)}(c) = 0$, but $g^{(n)}(c) \neq 0$, show that $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}$.

Pf: By Taylor's Thm, we have

$$\frac{f(x)}{g(x)} = \frac{\sum_{k=0}^{n-1} \frac{\cancel{f^{(k)}(c)}^{=0}}{k!}(x-c)^k + \frac{f^{(n)}(z_1)(x-c)^n}{n!}}{\sum_{k=0}^{n-1} \frac{\cancel{g^{(k)}(c)}^{=0}}{k!}(x-c)^k + \frac{g^{(n)}(z_2)(x-c)^n}{n!}}$$

some z_1 between x, c.

some z_2 between x, c.

$$= \frac{f^{(n)}(z_1)}{g^{(n)}(z_2)}.$$

So taking limit $x \to c$ $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f^{(n)}(z_1)}{g^{(n)}(z_2)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$

219: Show that the function $f(x) = x^3 - 2x - 5$ has a zero r in $[2, 2.2]$. If $x_1 = 2$, define (x_n) by Newton's method. Show that

$$|x_{n+1} - r| \leq \underline{(0.7)} |x_n - r|^2$$

and compute x_4.

Pf: $f(2) = -1 < 0$, $f(2.2) = 1.248 > 0$ ∴ so f has a zero in $[2, 2.2]$. using f cts & IVT

$|f'(x)| = |3x^2 - 2| \geq |3 \cdot 2^2 - 2| = 10.$

$|f''(x)| = |6x| \leq |6 \cdot 2.2| \leq 13.2.$

$K = \frac{M}{2m} = \frac{13.2}{20} = 0.66.$

So by Newton's method (thm), we have $|x_{n+1} - r| \leq (0.66)|x_n - r|^2.$

$f(x) = x\ln x.$

$x_1 = 2$

$$x_2 = 2 - \frac{f(2)}{f'(2)} = 2.1$$

$$x_3 = 2.1 - \frac{f(2.1)}{f'(2.1)} = \frac{11761}{5615} = 2.094568121104\ldots$$

$$x_4 = \frac{11761}{5615} - \frac{f\left(\frac{11761}{5615}\right)}{f'\left(\frac{11761}{5615}\right)} = 2.0945515\ldots$$